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Limited Communication and Incentive Compatibility

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1. Introduction

Two of the major themes of Leo Hurwicz's work are the construction of incentive-compatible mechanisms and the informational requirements of decentralized resource-allocation processes. In this chapter we consider a problem that arises because of the interaction of these concerns. We study the impact of limited communication possibilities on the design and performance of incentive-compatible mechanisms. The privacy of information and the conflicting objectives of the agents give rise to the need for incentive-compatible procedures. The complexity of information places some limits on the possibility of its full communication and utilization. Costs of transmission, storage, and information processing are among the factors that could cause a designer to limit the potential for information flows among agents. We model these constraints by taking a small-dimensional message space in a resource-allocation mechanism—too small to permit a full interchange of information even if this were incentive compatible.

The plan of this chapter can be viewed using the Mount and Reiter commutative diagram (Figure 12.1).

Usually, the choice of message space and the outcome function are not constrained when the mechanism is designed. We place some *a priori* restriction on the message space. Thus, necessarily, messages condense the private information into some summary statistics, and these statistics determine the outcome. The problem of incentive-compatible design of an im-

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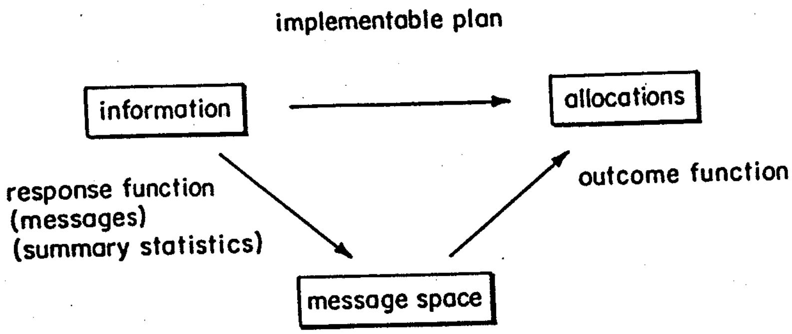

Figure 12.1. Mount and Reiter commutative diagram.

plementable plan under these restrictions encompasses both the choice of these statistics and the way they will be utilized.

The literature on incentive compatibility is now quite extensive. However, with only a few exceptions,[1] it is assumed that agents can transmit messages that are sufficiently detailed to describe fully all their private information. Even when this kind of full communication is possible, it is only under very special conditions that the differences in objectives can be mitigated by the use of an incentive-compatible resource-allocation mechanism capable of achieving a fully efficient outcome. When we impose constraints on information transmission, it is natural to expect a further departure from optimality.

There is a well-developed theory of the minimal size of communication network necessary to achieve any given resource-allocation plan[2] assuming complete cooperation of the agents. Particular attention is focused on plans that are fully efficient and on particular patterns of the dispersal of information, such as the competitive-exchange model[3] in which each agent knows

[1]For example, Groves and Ledyard (1977) requires only local information under the Nash solution, and the median-voter rule requires only the revelation of the agent's best outcome in the case of single-peaked preferences. Similarly, in the dynamic planning procedures (Drèze and de la Vallée Poussin 1971; Malinvaud 1971) only marginal rates of substitution along a path are transmitted. In this case full revelation of all privately held information was not necessary to implement the social optimum.

[2]See Hurwicz (1972), Mount and Reiter (1974) and Reiter (1974).

[3]Osana (1978) showed that the competitive mechanism uses the minimal number of dimensions possible if full efficiency is to be attained. Jordan (1982) showed that it is the only individually rational mechanism, with this dimensionality fixed, capable of achieving this performance.

only his own preferences and endowment. Recently,[4] answers to this minimality problem have been given for systems with incentive problems as well.

The theory of teams[5] has addressed the problem of limited communication possibilities where all agents had a common objective. Team theory has tried to compare different communication systems by evaluating their performance when each was most effectively utilized.

In this investigation we look at the problem of conflicting objectives and strategic behavior in the presence of limited communication possibilities. We characterize those resource-allocation plans that are implementable when the message space is smaller than the space of the agent's information. We discuss the loss due to the incentive-compatibility constraints experienced in addition to the costs of limited communication.

In the next section we set out the model used to study this question. Section 3 gives the general form of the constraints on resource allocation that result. Section 4 is devoted to the special case of a quadratic objective function and a communication system limited to the transmission of a single real number. In this instance, an explicit solution to the form of implementable plans can be given.

In Section 5 we consider the principal-agent problem in which the agent is described as in Section 4. The principal has a different quadratic objective than the agent. We solve the principal's problem, assuming normality of the parameters, with a one-dimensional and a two-dimensional message space, and both with and without the incentive-compatibility constraints caused by the difference in their objectives. We show that with a two-dimensional message space the presence of these constraints does not decrease the value of the principal's problem. However, with communication restricted to only a one-dimensional message, the incentive constraints become binding. In general, the principal is hurt by both the limited ability to receive information from the agent and by the agent's potential to distort private information. Finally, some numerical computations are offered through which the sensitivity of these results to the parameters is explored.

2. A General Model of Incentives and Limited Communication

We consider a model with one economic agent and a central resource-allocation unit. The decision taken by the central unit is a real vector $x \in R^n$.

[4]See Reichelstein (1980), where the lower bound on the dimensionality of the message space needed to attain optimality is given. In a converse vein, E. Green (1982) showed that, generically, an agent cannot be motivated to reveal any prespecified lower-dimension set of statistics of his observations.

[5]See Marschak and Radner (1972).

The vector $\theta \in R^m$ represents the parameters relevant to both players' objectives. It is assumed that the agent learns the value of θ and is then required to transmit a vector $\alpha \in R^l$. The value of x is chosen in a predetermined way as a function of α.

In addition to x and θ, we assume that the objectives of both players depend on their allocation of a transferable resource, for example, money. The quantity of this resource transferred from the center to the agent will be denoted $t \in R$. As in the case of x, t is determined by α.

To make the problem tractable, it is assumed that both players' objective functions are additively separable in the transferable resource. Thus, we write these objectives as

$$u(x,\theta) + t$$

for the agent, and

$$v(x,\theta) - t$$

for the central allocation unit. Moreover, u and v are assumed to be twice-continuously differentiable.

A *mechanism* for this model is a pair of functions

$$x: R^l \rightarrow R^n$$

$$t: R^l \rightarrow R$$

giving the values $x(\alpha)$, $t(\alpha)$ to be realized if the agent transmits α.

For simplicity, we restrict the analysis to twice-continuously differentiable mechanisms, that is, to mechanisms such that x and t are twice-continuously differentiable.

Given the mechanism (x,t), the agent chooses a *response rule*

$$\alpha: R^m \rightarrow R^l$$

giving the value of α associated to each $\theta \in R^m$. The *optimal response rule*, α^*, is the function of θ that for each $\theta \in R^m$ assigns the value $\alpha(\theta)$ maximizing $u(x(\alpha(\theta)),\theta) + t(\alpha(\theta))$.

If (x,t) is a mechanism for which α^* is an optimal response rule, the composition of x and α^* describes the decision chosen for each θ, namely, $x(\alpha^*(\theta))$. We will say that the resource allocation plan $z: R^m \rightarrow R^n$, $\theta \rightarrow z(\theta) = x \in R^n$ is *implementable* if there exists a mechanism (x,t) such that an optimal response α^* to (x,t) satisfies

$$z = x \circ \alpha^*.$$

In the next two sections we study the set of implementable resource-allocation plans and particularly the way in which implementable plans depend on the relationships among the dimensionalities of the decision space,

n, the parameter space, m, and the transmission space, l. It is clear that the set of implementable plans depends only on the agent's utility function.

In Section 5 we study how the central unit can select, within the set of implementable plans, the one that maximizes his own expected utility, subject to the constraint that the agent receives at least an expected utility of $\bar{u}$. This optimization problem is simplified by the assumed quasi linearity of the utility functions, which reduce it to the finding of that implementable plan that maximizes the expectation of $u + v$. Transfers are then adjusted to satisfy the individual rationality constraint.

3. Characterizing Implementable Plans

We now proceed to characterize the family of implementable plans. Imagine that an implementable plan z has been found and that z is implemented by the use of the mechanism (x,t), which induces an optimal response α^*. Note that the dimensionality of α may be smaller than the dimensionality of θ.

The plan $z(\cdot)$ is implemented by a mechanism (x,t) using strategy spaces R^m, which are different, in general, from the message spaces. From the revelation principle, we know that there exists a direct revelation mechanism $z(\theta)$, $s(\theta)$ that implements $z(\cdot)$ and that induces truthful revelation.

The proof of the revelation principle is actually straightforward, and we sketch it here for completeness. Let α^* be the optimal response rule of the agent when he faces the mechanism (x,t). Consider now the mechanism defined on R^m by

$$z(\theta) = x(\alpha^*(\theta))$$

$$s(\theta) = t(\alpha^*(\theta)).$$

To see that it induces truthful responses, suppose that θ' were a better response than the truth θ. Then,

$$u(x(\alpha^*(\theta')),\theta) + t(\alpha^*(\theta')) > u(x(\alpha^*(\theta)),\theta) + t(\alpha^*(\theta)).$$

But then, $\alpha^*(\theta') \neq \alpha^*(\theta)$ would be a better response for the agent faced with (x,t), a contradiction of the assumed optimality of α^*.

Therefore, a first set of constraints required from implementable plans is obtained by expressing the truth-telling constraints on $z(\cdot)$, $s(\cdot)$. The other constraints on z will arise solely from the fact that it must be written as the composition of x and α^*, defined over the domains R^l and R^m, respectively.

In order to express the truth-telling constraints on the composition of x and α^* that apply to the family of implementable plans z by virtue of the above argument, we write the agent's objective function as

$$u(z(\theta),\theta) + s(\theta) \tag{3.1}$$

where

$$z(\theta) = x(\alpha^*(\theta))$$
$$s(\theta) = t(\alpha^*(\theta)).$$

If the agent were responding to this mechanism truthfully, we would have the first-order conditions[6]

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial z_i}{\partial \theta_k} = -\frac{\partial s}{\partial \theta_k} \quad k = 1, \ldots, m, \tag{3.2}$$

and these would hold as identities in θ. One system of constraints on $z(\theta)$ is developed from the integrability of s, which implies (Young's theorem)

$$\frac{\partial}{\partial \theta_k}\frac{\partial s}{\partial \theta_{k'}} = \frac{\partial}{\partial \theta_{k'}}\frac{\partial s}{\partial \theta_k} \quad k, k' = 1, \ldots, m.$$

In the present case we have

$$\frac{\partial}{\partial \theta_{k'}}\left(\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial z_i}{\partial \theta_k}\right) = \sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial^2 z_i}{\partial \theta_k \partial \theta_{k'}} + \frac{\partial^2 u}{\partial x_i \partial \theta_{k'}}\frac{\partial z_i}{\partial \theta_k}$$
$$+ \sum_{i'=1}^{n}\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i \partial x_{i'}}\frac{\partial x_{i'}}{\partial \theta_{k'}}\frac{\partial x_i}{\partial \theta_k}.$$

Interchanging the roles of k and k' and equating these expressions produces the relations

$$\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i \partial \theta_{k'}}\frac{\partial z_i}{\partial \theta_k} = \sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i \partial \theta_k}\frac{\partial z_i}{\partial \theta_{k'}}, \quad 1 \le k < k' \le m. \tag{3.3}$$

In addition, implementability is constrained by the second-order conditions of the agent's optimization problem (3.1). Writing

$$V_k(\theta,\hat{\theta}) \equiv \sum_i \frac{\partial u}{\partial x_i}(z_i(\theta),\hat{\theta})\left(\frac{\partial z_i}{\partial \theta_k}(\theta)\right) + \frac{\partial s}{\partial \theta_k} \quad k = 1, \ldots, m$$

and

$$V(\theta,\hat{\theta}) = (V_1(\theta,\hat{\theta}), \ldots, V_m(\theta,\hat{\theta})),$$

the first-order conditions (3.2) can be rewritten

$$V(\theta,\hat{\theta}) = 0 \quad \text{for } \theta = \hat{\theta}. \tag{3.4}$$

[6]See Laffont and Maskin (1980).

The second-order necessary conditions are expressed as

$$\frac{d}{d\theta} V(\theta,\hat{\theta}) \quad \text{negative definite at } \theta = \hat{\theta}.$$

Because (3.4) is an identity, we know that

$$\frac{d}{d\theta} V + \frac{d}{d\hat{\theta}} V = 0 \quad \text{at } \theta = \hat{\theta}$$

so that an equivalent necessary condition is

$$-\frac{d}{d\hat{\theta}} V \quad \text{negative definite at } \theta = \hat{\theta}. \tag{3.5}$$

In order that $\theta = \hat{\theta}$ be the global maximum, it is sufficient to show, in addition, that

$$-\frac{d}{d\theta} V(\theta,\hat{\theta}) \quad \text{negative definite for all } \theta,\hat{\theta}. \tag{3.6}$$

These second-order conditions will not be used presently in the general case. They will provide some simple additional restrictions on the form of implementable plans in the special cases for which explicit solutions can be derived (see Section 4).

The interesting nature of constraints imposed by the first-order conditions derives from the relationship between the number of components to be controlled, n, and the dimensionality of the underlying parameter space, m. For $m = 1$, these constraints are vacuous, of course. The limitations on implementability then arise only from the second-order conditions on the agent's optimal behavior.[7] For $n = 1$ and $m = 2$, we have a single condition:

$$\frac{\partial^2 u}{\partial x \partial \theta_1} \frac{\partial z}{\partial \theta_2} = \frac{\partial^2 u}{\partial x \partial \theta_2} \frac{\partial z}{\partial \theta_1}$$

or

$$\frac{\dfrac{\partial z}{\partial \theta_1}}{\dfrac{\partial z}{\partial \theta_2}} = \frac{\dfrac{\partial}{\partial \theta_1} \dfrac{\partial u}{\partial x}}{\dfrac{\partial}{\partial \theta_2} \dfrac{\partial u}{\partial x}}. \tag{3.7}$$

[7]Assuming $d^2\mu/d\theta dx > 0$, the second-order condition is just $dz/d\theta \geq 0$. (See Laffont and Maskin 1980.) We neglect here the problem of the integrability of the differential equations.

In more familiar terms, this means that if x is implemented at (θ_1,θ_2), then the locus in the (θ_1,θ_2) plane along which x is constant must be precisely the locus along which $\partial u/\partial x$ is constant.

In general, for $n = 1$, and $m \geqq 2$, the solutions to the generalization of (3.7) are precisely those functions $z(\theta)$ of the form $z(\theta) = \phi(\partial u/\partial x)$ for an arbitrary function ϕ.

Compared to the lower-dimensional case $m = 1$, this gives us a rather pessimistic conclusion about the possibility of controlling an agent's behavior. For example, in (3.1) if the agent's utility depends only on θ_2, whereas both parameters are relevant to the principal, there is no way to enforce a plan that uses the agent's knowledge of θ_1, as $\partial z/\partial \theta_1$ must be identically zero.

As we will see below, the problem of limited channels for information transmission does not complicate matters any further when $n = 1$. Therefore, before leaving the incentive-compatibility constraints, it is useful to consider the case $n = 2$, $m = 2$, for here there will be an interplay between the two kinds of constraints we are examining.

In this case, the system (3.3) is reduced to

$$\frac{\partial^2 u}{\partial x_1 \partial \theta_2}\frac{\partial z_1}{\partial \theta_1} + \frac{\partial^2 u}{\partial x_2 \partial \theta_2}\frac{\partial z_2}{\partial \theta_1} = \frac{\partial^2 u}{\partial x_1 \partial \theta_1}\frac{\partial z_1}{\partial \theta_2} + \frac{\partial^2 u}{\partial x_2 \partial \theta_1}\frac{\partial z_2}{\partial \theta_2}. \tag{3.8}$$

This is a single partial differential equation in the two functions $z_1(\theta_1,\theta_2)$ and $z_2(\theta_1,\theta_2)$. One form of solution is to make z_1 depend on $\partial u/\partial x_1$ and z_2 on $\partial u/\partial x_2$, paralleling the family of solutions for the case $n = 1$ above. In general, however, there are many other solutions to (3.8), as we will see in the next section. We will be interested in which of the solutions to (3.8) is compatible with the constraints induced by limited information transmission capabilities.

We now address the question of the nature of the constraints placed on an implementable plan $z(\theta)$ owing to the fact that $z = x \circ \alpha$ where α: $R^m \rightarrow R^l$ and x: $R^l \rightarrow R^n$. Assuming that $x(\cdot)$ is differentiable, the range of z is limited to a l-dimensional subset[8] of R^n, and the matrix

$$Z = \frac{\partial z_i}{\partial \theta_k} \quad \begin{array}{l} i = 1, \ldots, n \\ k = 1, \ldots, m \end{array} \tag{3.9}$$

can be at most of rank l. This explains why these constraints are interesting

[8]Without a smoothness assumption of this sort, it would be possible to "code" two real numbers into one, for example, by the use of a space-filling curve. See Hurwicz (1972) and Mount and Reiter (1974) for further discussion.

only if $l < \min(n,m)$. This rank condition represents the constraints imposed on implementable resource-allocation plans owing to limitations on information processing as described in the introduction. Thus, the problem we pose is to characterize the nature of the solutions to the incentive-compatibility constraints that obey this information constraint as well. We will see, at least in the special cases examined in the next section, that the conjunction of these conditions is very restrictive.

4. The Quadratic Case: with $n = m = 2$, $l = 1$

In this section,[9] we solve for the twice-continuously differentiable solutions of the system (3.8) (3.9) constraining the implementable plans under limited communication in the special case where

$$u(x_1,x_2,\theta_1,\theta_2) = a_1\theta_1x_1 + a_2\theta_2x_2 - 1/2\,x_1^2 - 1/2\,x_2^2 \quad a_1, a_2 > 0. \tag{4.1}$$

Then, (3.8) takes the form of a system of linear partial differential equations with constant coefficients. Explicit solutions are obtainable.

We can write the information constraints (3.9) as

$$\frac{\dfrac{\partial z_1}{\partial \theta_1}}{\dfrac{\partial z_1}{\partial \theta_2}} = \frac{\dfrac{\partial z_2}{\partial \theta_1}}{\dfrac{\partial z_2}{\partial \theta_2}} \tag{4.2}$$

when both ratios are well defined, or as

$$\frac{\partial z_1}{\partial \theta_2} = \frac{\partial z_2}{\partial \theta_2} = 0 \tag{4.3}$$

otherwise.

For the utility function (4.1), the incentive constraints reduce to

$$a_1\frac{\partial z_1}{\partial \theta_2} = a_2\frac{\partial z_2}{\partial \theta_1}. \tag{4.4}$$

We first describe the solutions to (4.4) ignoring any problems of information transmission. Then we will impose (4.2) or (4.3) and show how the set of implementable plans is restricted.

Choose $z_2(\theta_1,\theta_2)$ arbitrarily and integrate

[9] See Appendix A for the characterization of the incentive-compatible mechanisms when $n = m = 3$ and $l = 1$.

$$\frac{\partial z_1}{\partial \theta_2} = \frac{a_2}{a_1}\frac{\partial z_2}{\partial \theta_1}$$

using the initial condition $z_1(\theta_1,0) = \phi(\theta_1)$, where ϕ is an arbitrary function. This yields

$$z_1(\theta_1,\theta_2) = \frac{a_2}{a_1}\int_0^{\theta_2} z_2(\theta_1,\xi)d\xi + \phi(\theta_1). \tag{4.5}$$

Therefore, the (first-order) incentive constraints limit the choice of $z_1(\theta_1,\theta_2)$, $z_2(\theta_1,\theta_2)$ to the choice of one arbitrary function of two variables and another arbitrary function of one variable, instead of the full family of pairs of functions of two variables that would be available were incentives not a problem.

Let us first treat the case of (4.2). Take $z_2(\theta_1,\theta_2)$ arbitrary and use the incentive constraints to write

$$\frac{\partial z_1}{\partial \theta_2} = \frac{a_2}{a_1}\frac{\partial z_2}{\partial \theta_1}$$

$$\frac{\partial z_1}{\partial \theta_1} = \frac{a_2}{a_1}\left(\left(\frac{\partial z_2}{\partial \theta_1}\right)^2 \Big/ \frac{\partial z_2}{\partial \theta_2}\right). \tag{4.6}$$

In order for this function z_2 to define a function z_1, (4.6) must satisfy the integrability condition $\partial/\partial\theta_1\ \partial z_1/\partial\theta_2 = \partial/\partial\theta_2\ \partial z_1/\partial\theta_1$. Equating these expressions, we have

$$0 = \frac{\partial^2 z_2}{\partial \theta_1^2}\left(\frac{\partial z_2}{\partial \theta_2}\right)^2 - 2\frac{\partial z_2}{\partial \theta_2}\frac{\partial z_2}{\partial \theta_1}\frac{\partial^2 z_2}{\partial \theta_1 \partial \theta_2} + \frac{\partial^2 z_2}{\partial \theta_2^2}\left(\frac{\partial z_2}{\partial \theta_1}\right)^2. \tag{4.7}$$

The right-hand side of this equation is just the Gaussian curvature of a level curve of a locus where z_2 is constant in the (θ_1,θ_2) plane. Thus, all such level curves are linear, and we have

$$z_2 = \psi(\lambda\theta_1 + \mu\theta_2) \tag{4.8}$$

for some parameters λ, μ with $\mu \neq 0$ and an arbitrary function ψ with $\psi' \neq 0$.

From the informational constraints (4.2), it follows that

$$z_1 = \bar{\psi}(\lambda\theta_1 + \mu\theta_2), \tag{4.9}$$

where $\bar{\psi}' \neq 0$, because without loss of generality we can take

$$\alpha = \lambda\theta_1 + \mu\theta_2.$$

Then, using (4.6), (4.8), and (4.9), we have kets. For example, we see from above that if $K = 0$, $\lambda/\mu = \alpha_1/\alpha_2$, the baskets that are achievable involve the same quantities of goods 1 and 2.

5. Loss Due to the Interaction of Incentive and Communication Constraints

We consider a principal-agent problem in which the principal's evaluation of the decision is given by

$$v(x_1,x_2,\theta_1,\theta_2) = b_1\theta_1x_1 + b_2\theta_2x_2 - 1/2x_1^2 - 1/2x_2^2$$

and the agent's evaluation is, as in (4.1),

$$u(x_1,x_2,\theta_1,\theta_2) = a_1\theta_1x_1 + a_2\theta_2x_2 - 1/2x_1^2 - 1/2x_2^2$$

where a_1, a_2, b_1, b_2 are all positive. The principal believes that (θ_1,θ_2) are jointly normally distributed with mean zero and covariance matrix Σ, so that their density function is

$$g(\theta_1,\theta_2) = \frac{1}{2\pi|\sum|^{1/2}} e^{-1/2((\theta_1,\theta_2)\Sigma(\theta_1,\theta_2)')}$$

where

$$\sum = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

The principal designs the mechanism (x,t) as in Section 3 so as to maximize $E(v - t)$ under the constraint that the agent's utility is at least $\bar{u}$,

$$E(u + t) \geqq \bar{u}$$

where the expectation is taken with respect to the density g. Because of the linearity of the objective functions in the transfer t, the x function that solves this problem is the maximizer of $E(u + v)$. Therefore, we consider the problem

$$\max \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [c_1\theta_1x_1 + c_2\theta_2x_2 - 1/2x_1^2 - 1/2x_2^2]g(\theta_1,\theta_2)d\theta_1d\theta_2 \quad (5.1)$$

where

$$c_1 = \frac{a_1 + b_1}{2}, \; c_2 = \frac{a_2 + b_2}{2}.$$

In the next part of this section we consider this problem under various forms of the incentive and communication constraints. We examine the impact of the incentive constraints on the value of the principal's problem for different dimensionalities of the message space. We show that the incentive constraint is not binding when full dimensional communication is possible, but that it is binding, in general, when the message space has dimensionality one.

The unconstrained solution of (5.1) is

$$x_1^* = c_1\theta_1$$

$$x_2^* = c_2\theta_2. \tag{5.2}$$

With a two-dimensional message space, the incentive constraints are (see (4.4) and (3.6))

$$\text{(i)} \qquad a_1 \frac{\partial x_1}{\partial \theta_2} = a_2 \frac{\partial x_2}{\partial \theta_1}$$

$$\text{(ii)} \qquad \frac{\partial x_1}{\partial \theta_1} \geqq 0 \quad \frac{\partial x_2}{\partial \theta_2} \geqq 0 \quad \frac{\partial x_1}{\partial \theta_1}\frac{\partial x_2}{\partial \theta_2} - \frac{\partial x_1}{\partial \theta_2}\frac{\partial x_2}{\partial \theta_1} \geq 0. \tag{5.3}$$

These constraints are fulfilled by (5.2)[11] and there is no loss to the principal. The value of this problem, that is, the value of $E(u + v)$, is

$$V = \frac{c_1^2\sigma_1^2 + c_2^2\sigma_2^2}{2} - \bar{v}. \tag{5.4}$$

We now consider the problem with a one-dimensional message space but without incentive-compatibility constraints. Because of the quadratic objective and the normality of (θ_1,θ_2), this problem is equivalent to finding the straight line that minimizes the expected distance from the optimal actions (5.2). We can see that the solution takes the form

$$x_1(\theta_1,\theta_2) = \frac{c_1\theta_1 + \beta c_2\theta_2}{1 + \beta^2}$$

$$x_2(\theta_1,\theta_2) = \frac{c_1\theta_1 + \beta c_2\theta_2}{1 + \beta^2}. \tag{5.5}$$

We then choose β to solve (5.1), using (5.5) to determine the actions.

[11]In general, the first-best will not satisfy the incentive-compatibility restrictions, even with full communication possible. It happens in this example because of the absence of an interaction between x_1 and x_2 in the central unit's objective. This results in (5.2), where x_1^* is independent of θ_2 and x_2^* is independent of θ_1, so that (5.3)(i) is satisfied with both terms identically zero.

Lengthy computations give two candidates for the optimal β:

$$\beta = \frac{\delta \mp \sqrt{\delta^2 + 4}}{2} \tag{5.6}$$

where

$$\delta = \frac{c_2^2\sigma_2^2 - c_1^2\sigma_1^2}{c_1c_2\sigma_{12}}.$$

The value of the problem is

$$\frac{1}{2(1 + \beta^2)} [c_1^2\sigma_1^2 + 2c_1c_2\sigma_{12}\beta + c_2^2\sigma_2^2\beta^2] \tag{5.7}$$

where the choice of the two values of β given in (5.6) is made so as to maximize this expression.

In the special case $c_1 = c_2$ and $\sigma_1^2 = \sigma_2^2$, the value of the principal's problem is

$$V = \frac{c_1}{2}(\sigma_1^2 + |\sigma_{12}|) - \bar{v}. \tag{5.8}$$

Comparing (5.4) and (5.8), we can see that as the correlation of θ_1 and θ_2 approaches unity ($\sigma_{12} \to \sigma_1^2$ in this special case), the decreased expected value of the principal's objective function, owing to the smaller dimensional message space, becomes progressively less severe.

We will now examine the problem imposing the incentive-compatibility requirements developed in Section 4 for the case of a one-dimensional channel.

To simplify notation, we set $a_1 = a_2 = 1$. Substituting (4.10) into (5.1), we obtain the objective function

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{c_1\theta_1\left[\frac{\lambda}{\mu}\psi + K\right] + c_2\theta_2\psi - 1/2\left[K^2 + \left(1 + \frac{\lambda^2}{\mu^2}\right)\psi^2 + 2K\frac{\lambda}{\mu}\psi\right]\right\}g(\theta_1,\theta_2)d\theta_1 d\theta_2. \tag{5.9}$$

This is to be maximized with respect to λ, μ, K, and the function $\psi(\cdot)$, subject to $\mu > 0$ and $\psi' > 0$.[12] (The degenerate case of (4.11), which we

[12] We restrict here the optimization to differentiable solutions of the incentive constraints.

do not treat explicitly, corresponds to the case of $\mu = 0$.) Several steps of this maximization are carried out in appendix B. Then, the optimization problem is reduced to

$$\max_{(\lambda,\mu)} \frac{[c_2(\sigma_2^2\mu^2 + 2\sigma_{12}\lambda\mu + \sigma_1^2\lambda^2) + \lambda(c_1 - c_2)(\sigma_{12}\mu + \sigma_1^2\lambda)]^2}{2(\lambda^2 + \mu^2)(\sigma_2^2\mu^2 + 2\sigma_{12}\lambda\mu + \sigma_1^2\lambda^2)}. \quad (5.10)$$

The difference between the maximized values of (5.7) and (5.10) measures the loss due to the presence of incentive-compatibility constraints when the message space is one-dimensional. Table 12.1 shows how these values change for fixed c_1, c_2, σ_1^2, and σ_2^2 when the correlation between θ_1 and θ_2 varies.

We can interpret variations in σ_{12} as a parametric way of representing the amount of information available to the agent relative to the amount that can be transmitted through this channel. (This can be made precise by observing

Table 12.1. Loss Due to Incentive Under Various Parameter Values

$\sigma_1 = \sigma_2 = 1$

$c_1 = 1 \quad c_2 = 2$

σ_{12}	Value without Incentive Constraint	β	Value with Incentive Constraint	λ	μ
0.00	2.0000	∞	2.0000	0.00	1.00
0.05	2.0017	30.00	2.0012	0.15	6.00
0.10	2.0066	15.06	2.0050	0.40	7.95
0.15	2.0149	10.09	2.0113	0.25	3.30
0.20	2.0262	7.63	2.0202	0.80	7.85
0.25	2.0406	6.16	2.0318	0.70	5.45
0.30	2.0578	5.19	2.0461	0.35	2.25
0.35	2.0777	4.51	2.0631	1.65	9.00
0.40	2.1000	4.00	2.0829	1.30	6.15
0.45	2.1246	3.61	2.1055	1.15	4.80
0.50	2.1514	3.30	2.1308	2.65	9.90
0.55	2.1800	3.05	2.1587	.65	2.20
0.60	2.2105	2.85	2.1890	.50	1.55
0.65	2.2425	2.68	2.2217	1.90	5.45
0.70	2.2759	2.54	2.2565	3.55	9.50
0.75	2.3107	2.41	2.2932	1.85	4.65
0.80	2.3466	2.31	2.3317	2.65	6.30
0.85	2.3836	2.22	2.3718	4.40	9.95
0.90	2.4215	2.13	2.4133	3.40	7.35
0.95	2.4604	2.06	2.4561	4.65	9.65
1.00	2.5000	2.00	2.5000	0.55	1.10

that the mutual information [in the sense of Shannon, 1948] between (θ_1,θ_2)[13] and α is increasing in σ_{12}.) Note that when $\sigma_{12} = 1$, the problem is effectively one-dimensional. In this case there is no loss because any (monotonic) choice of actions can be implemented with this message space. Observe also that when $c_1 = c_2$ (or, more generally, $c_1/c_2 = a_1/a_2$), the incentive-compatibility constraints are not binding even with a one-dimensional message space. This can be seen by noting that (5.5) is of the form of (4.10).[14]

When $\sigma_{12} = 0$, the principal would select the summary statistic $\alpha = \theta_2$ and would utilize it by setting $x_1 = K$, $x_2 = b_2\theta_2$. As this is incentive compatible, there is no loss due to incentive effects. More generally, for intermediate values of σ_{12}, the principal's solution of the problem with a one-dimensional channel requires a choice of statistic and a decision rule that are in conflict with the incentive-compatibility constraint. Therefore, the loss due to the incentive constraints will not be monotonic in the amount of information the agent has relative to what he could transmit.

Appendix A

We characterize the incentive-compatible mechanisms when $n = m = 3$ and $l = 1$.

We retain the assumption that the utility function is quadratic and additively separable:

$$u(x_1,x_2,x_3,\theta_1,\theta_2,\theta_3) = a_1\theta_1x_1 + a_2\theta_2x_2 + a_3\theta_3x_3 - 1/2x_1^2 - 1/2x_2^2 - 1/2x_3^2$$

where

$$a_1 > 0, a_2 > 0, a_3 > 0.$$

We will show that all incentive-compatible solutions that can be implemented with one channel of communication take the same form as in the two-variable cases—namely, the channel must be linear, and, apart from additive constants, there is only the flexibility to control one of the quantities. The others are functionally dependent on it and on the choice of the linear channel.

There are three incentive constraints.

[13]The mutual information of θ_1 and θ_2 is the entropy of θ_1 minus the conditional entropy of θ_1 given θ_2. It represents the mean amount of information that knowledge of the value taken by θ_2 supplies about the value taken by θ_1 (see Berger 1971).

[14]In cases where the lack of symmetry of the distribution of θ_1 and θ_2 causes the optimum of the problem with $l = 1$ and no incentive constraint to be a nonlinear function of θ_1 and θ_2, the incentive constraints (for an agent with a quadratic objective) will be binding. The incentive constraints allow only the transmission of linear functions, even though the two players could agree on how to utilize a nonlinear statistic if it could be transmitted. This feature is not present above because of the joint normality of (θ_1,θ_2).

$$a_1\left(\frac{\partial z_1}{\partial \theta_2}\right) = a_2\left(\frac{\partial z_2}{\partial \theta_1}\right)$$

$$a_2\left(\frac{\partial z_2}{\partial \theta_3}\right) = a_3\left(\frac{\partial z_3}{\partial \theta_2}\right)$$

$$a_3\left(\frac{\partial z_3}{\partial \theta_1}\right) = a_1\left(\frac{\partial z_1}{\partial \theta_3}\right).$$

Now we impose the informational constraints from the fact that the communication takes the form of a unidimensional function $\alpha(\theta_1,\theta_2,\theta_3)$:

$$z_1 = z_1(\alpha(\theta_1,\theta_2,\theta_3))$$

$$z_2 = z_2(\alpha(\theta_1,\theta_2,\theta_3))$$

$$z_3 = z_3(\alpha(\theta_1,\theta_2,\theta_3)). \tag{A.1}$$

If we consider θ_3 fixed, and examine the incentive- and information-compatible solutions for z_1 and z_2 as functions of θ_1, θ_2, we can use the results of the previous section to write:[15]

$$z_1 = \frac{a_2}{a_1}\frac{\lambda}{\mu}\psi(f(\lambda\theta_1 + \mu\theta_2,\theta_3)) + K_{12}$$

$$z_2 = \psi(f(\lambda\theta_1 + \mu\theta_2,\theta_3)).$$

Likewise, for each θ_1, we obtain

$$z_2 = \frac{a_3}{a_2}\frac{\eta}{\nu}\phi(g(\eta\theta_2 + \nu\theta_3,\theta_1)) + K_{23}$$

$$z_3 = \phi(g(\eta\theta_2 + \nu\theta_3,\theta_1)),$$

and for each θ_2, we obtain

$$z_1 = \chi(h(\tau\theta_3 + \sigma\theta_1,\theta_2))$$

$$z_3 = \frac{a_1}{a_3}\frac{\tau}{\sigma}\chi(h(\tau\theta_3 + \sigma\theta_1,\theta_2)) + K_{13}.$$

The functions f, g, and h represent the value of α taken on for each θ_1, θ_2,

[15]We examine only the strictly monotonic solutions in both variables because the others are obtained by symmetry for limiting values of the parameters λ and μ.

θ_3. In each of these three cases, our earlier results have limited the allowable functional dependences. We can show that the functional dependence of α on θ_1, θ_2, θ_3 is linear, by equating f, g, and h as identities:

$$f(\alpha\theta_1 + \mu\theta_2, \theta_3) = g(\eta\theta_2 + \nu\theta_3, \theta_1) = h(\tau\theta_3 + \sigma\theta_1, \theta_2).$$

Differentiating with respect to θ_1 and θ_3, we obtain

$$\lambda f_1 = \sigma h_1$$

$$f_2 = \tau h_1$$

where subscripts indicate partial differentiation. Summarizing, $f_1 = \sigma/\lambda\tau f_2$. Thus f must depend on a linear combination of its arguments, which correspond to a linear combination of θ_1, θ_2, and θ_3. From (A.1), g and h are functions of the same linear combination.

Let

$$\alpha(\theta_1, \theta_2, \theta_3) = \alpha(c_1\theta_1 + c_2\theta_2 + c_3\theta_3)$$

be the common function of this linear dependence.

We now rewrite the system of six equations using this structure for α, f, g, and h, and develop relationships among the parameters and functions that are, up to now, constrained. Without loss of generality, the function α can be suppressed into the functions ψ, ϕ, and χ. Thus,

$$z_1 = \frac{a_2}{a_1}\frac{c_1}{c_2}\,\psi(c_1\theta_1 + c_2\theta_2 + c_3\theta_3) + K_{12} \tag{A.2}$$

$$z_1 = \chi(c_1\theta_1 + c_2\theta_2 + c_3\theta_3) \tag{A.3}$$

and

$$z_2 = \psi(c_1\theta_1 + c_2\theta_2 + c_3\theta_3) \tag{A.4}$$

$$z_2 = \frac{a_3}{a_2}\frac{c_2}{c_3}\,\phi(c_1\theta_1 + c_2\theta_2 + c_3\theta_3) + K_{23} \tag{A.5}$$

and

$$z_3 = \phi(c_1\theta_1 + c_2\theta_2 + c_3\theta_3) \tag{A.6}$$

$$z_3 = \frac{a_1}{a_3}\frac{c_3}{c_1}\,\chi(c_1\theta_1 + c_2\theta_2 + c_3\theta_3) + K_{13}. \tag{A.7}$$

Given an arbitrary function ψ, the function is determined by equating (A.2) to (A.3). Then, substituting into (A.7) and equating it to (A.6), we have an expression for ϕ in terms of ψ:

$$\phi = \frac{c_3}{c_2}\frac{a_2}{a_3}\psi + \frac{a_1 c_3}{a_3 c_1} K_{12} + K_{13}.$$

Substituting into (A.5) and equating the result to (A.4) yields an additional constraint on the parameters given by

$$0 = \frac{c_2}{c_1}\frac{a_1}{a_2} K_{12} + \frac{a_3}{a_2}\frac{c_2}{c_3} K_{13} + K_{23}$$

confirming the existence of only two independent additive constants. Summarizing this result, the set of implementable plans is given by

$$z_1 = \frac{a_2}{a_1}\frac{c_1}{c_2}\psi(c_1\theta_1 + c_2\theta_2 + c_3\theta_3) + K_{12}$$

$$z_2 = \psi(c_1\theta_1 + c_2\theta_2 + c_3\theta_3)$$

$$z_3 = \frac{a_2 c_3}{a_3 c_2}\psi(c_1\theta_1 + c_2\theta_2 + c_3\theta_3) + \frac{a_1 c_3}{a_3 c_1} K_{12} + K_{13}.$$

All implementable plans can be determined by five constants—a_1, c_2, c_3, K_{12}, and K_{13}—and a single arbitrary function of one variable—ψ.

As in the case of two variables, the only binding second-order conditions imply that $\psi' \geqq 0$, $c_2 \geqq 0$.

Appendix B

To simplify notation, we define

$$\Delta = \sigma_1^2\sigma_2^2 - \sigma_{12}^2$$

$$a = \frac{\sigma_2^2}{\Delta};\ b = \frac{\sigma_{12}}{\Delta};\ c = \frac{\sigma_1^2}{\Delta};\ \bar{k} = \frac{\Delta^{-1/2}}{2\pi}$$

$$\bar{k} = \frac{1}{2\pi\Delta}.$$

We must maximize with respect to λ, μ, K, $\psi(\cdot)$ the integral

$$I = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[c_1\theta_1\left(\frac{\lambda}{\mu}\psi + K\right) + c_2\theta_2\psi - 1/2\left(K^2 + \left(1 + \frac{\lambda^2}{\mu^2}\right)\psi^2 + 2K\frac{\lambda}{\mu}\psi\right)\right]g(\theta_1,\theta_2)d\theta_1 d\theta_2.$$

The change of variables $x = \lambda\theta_1 + \mu\theta_2$; $z = \theta_1$ yields

$$I = \int_{-\infty}^{+\infty} \frac{\bar{k}2\pi}{\mu\alpha^{1/2}} e^{\gamma-\beta^2/\alpha} \int_{-\infty}^{+\infty} \left\{ \left[c_1 K + \frac{\lambda}{\mu}(c_1 - c_2)\psi \right] z + (c_2 x - K\lambda)\frac{\psi}{\mu} - 1/2K^2 - 1/2\psi^2\left(1 + \frac{\lambda^2}{\mu^2}\right) \right\} \frac{\alpha^{1/2}}{2\pi} e^{-1/2(z-\beta/\alpha)^2/1/\alpha} dz\, dx$$

where

$$\alpha = a + \frac{2b\lambda}{\mu} + c\frac{\lambda^2}{\mu^2};\ \beta = \frac{b}{\mu} + c\frac{\lambda}{\mu^2};\ \gamma = \frac{cx^2}{\mu}.$$

Integrating with respect to z gives

$$I = \frac{\bar{k}}{\mu} \int_{-\infty}^{+\infty} \frac{2\pi}{\mu} e^{\gamma-\beta^2/\alpha} \left\{ \left[c_1 K + \frac{\lambda}{\mu}(c_1 - c_2)\psi \right] \frac{\beta}{\alpha} + (c_2 x - K\lambda)\frac{\psi}{\mu} - 1/2K^2 - 1/2\psi^2\left(1 + \frac{\lambda^2}{\mu^2}\right) \right\} dx.$$

The maximization with respect to ψ gives (Euler equation)

$$\psi(x) = \frac{1}{\lambda^2 + \mu^2} \left\{ \left(c_2\mu + \frac{\lambda\mu(c_1 - c_2)(b\mu + c\lambda)}{a^2\mu^2 + 2b\lambda\mu + c\lambda^2} \right) x - K\lambda\mu \right\}.$$

Substituting this expression into I, we now integrate with respect to x and find

$$I = \frac{\bar{k}}{\mu} \cdot \frac{2\pi}{\alpha^{1/2}} \cdot 2\pi \frac{(a\mu^2 + 2b\lambda\mu + c\lambda)^{1/2}}{(ac - b^2)^{1/2}} \cdot \left\{ -1/2K^2 + \frac{\lambda^2}{2(\lambda^2 + \mu^2)} K + \Delta \frac{(a\mu^2 + 2b\lambda\mu + c\lambda^2)}{2(\lambda^2 + \mu^2)} \left(b_2 + \frac{\lambda(b_1 - b_2)(b\mu + c\lambda)}{a\mu^2 + 2b\lambda\mu + c\lambda^2} \right)^2 \right\}.$$

The maximization with respect to K gives $k = 0$, and the problem is reduced to (5.10).

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